Comments on Ilya Tirdatov's paper "Managing Projects Involving External Threats"

In a recent issue on the American Society for the Advancement of Project Management's web site articles, www.asapm.org, a paper was published describing an important concept. In "Managing Projects Involving External Threats," Ilya Tirdatov describes the impact of the likelihood of an external event could occur that would negatively impact the success probability of a project. Such an event is neither expected nor predictable. The suggestion from the paper is that the sooner the project – or any subtask of the project – is completed, the less the likelihood of an external event impacts the project.

Although this statement seems obvious, it turns out to be insufficient for practical use. There is an important underlying concept not presented in the paper. In fairness to the author, the equation provided — $P_S = T^{-1} * 100$ — is stated to be a simple formulation of the problem and not representative of real situations.

This equation states there is a simple linear relationship between the passage of time and the probability of something bad happening. In fact the relationship between the passage of time and probability of an external event is "not" linear as stated above, but it is exponential based on the Poisson distribution. [1]

Exponential Probability

The exponential probability distribution is the most common distribution encountered in probability models, since it describes accurately most of the real life aspects of probability based processes. The probability density function (*pdf*), Cumulative Distribution Function (*CDF*), reliability function ($R(t)$), and hazard (event rate) function ($z(t)$) of the exponential distribution are expressed by the following:

$$pdf = f(t) = \lambda e^{-\lambda t} \tag{1.1}$$

$$CDF = F(t) = 1 - e^{-\lambda t} \tag{1.2}$$

$$\text{Reliability} = R(t) = e^{-\lambda t} \tag{1.3}$$

[1] One of the first observations of the Poisson process was that it properly represented the number of French cavalry soldiers that dies as a result of being kicked by a horse. *Rescherches sur la Probabilite des Jugements en Matiere Criminelle et en Matiere Civile, Precedees des Regles Generales du Calcul des Probabilities*, Simeon D. Poisson, 1837.

$$\text{Hazard Function} = z(t) = \lambda \tag{1.4}$$

Poisson Distributions

In modeling the expectation functions associated with actual processes, several simplifying assumptions can be made to render the resulting mathematics tractable. These assumptions do not reduce the applicability of the resulting models to real–world phenomenon. One simplifying assumption is that the random variables associated with the presence of an external event have exponential probability distributions.

The property of the exponential distribution that makes it easy to analyze is that it does not age with time – that is as time passes the probability of the event occurring remains the same. If the lifetime of a process is exponentially distributed, after some amount of time, the probability of event occurring is assumed to be *good as new*. Formally, this property states that the random variable X is *memoryless*, if the expression $P\left\{X > s+t \middle| X > t\right\} = P\left\{X > s\right\}$ is valid for all $s,\ t \geq 0$. If the random variable X is the lifetime of some item, then the probability that the item is functional at time $s+t$, given that it survived to time *t*, is the same as the initial probability that is was functional at time *s*. If the occurrence of an event is functional at time *t*, then the distribution of the remaining amount of time for the event to occur is the same as the original lifetime distribution. The event does not "remember" that it had not occurred for a time *t*.

This property is equivalent to the expression $\dfrac{P\left\{X > s+t,\, X > t\right\}}{P\left\{X > t\right\}} = P\left\{X > s\right\}$ or $P\left\{X > s+t\right\} = P\left\{X > s\right\} P\left\{X > t\right\}$. Since the form of this expression is satisfied when the random variable X is exponentially distributed (since $e^{-\lambda(s+t)} = e^{-\lambda s} e^{-\lambda t}$), it follows that exponentially distributed random variables are *memoryless*. The recognition of this property is vital to the understanding of the models of external events from unknown distributions. If the underlying event process is not memoryless, than the exponential distribution model is not valid. Determining the underlying distribution of external events is likely to be difficult. In order to proceed with a useful analysis of the impact of external events on a project, the assumption of a Poisson distribution can be used.

The exponential probability distributions and the related Poisson processes used in the reliability models are formally based on the assumptions shown in Table 1.

- Events occur completely randomly and are independent of any previous event. A single event does not provide any information regarding the time of the next event.
- The probability of an event during any interval of time $\left[0, t\right]$ is proportional to the length of the interval, with a constant of proportionality λ. The longer one waits the more likely it is an event will occur.

Table 1 – Assumptions regarding the behavior of a random process that generate events following the Poisson probability distribution function.

An expression describing the random processes in Table 1 results from the *Poisson Theorem* which states that the probability of an event A occurring k times in n trials is approximately,

$$\frac{n(n-1)\cdots(n-k+1)}{1\cdot 2\cdots k}p^k q^{n-k}, \tag{1.5}$$

where $p = P\{A\}$ is the probability of an event A occurring in a single trial and $q = 1 - p$. This approximation is valid when $n \to \infty, \quad p \to 0$ and the product $n \cdot p$ remains finite. It should be noted that a large number of *different* trials of independent systems is needed for this condition to hold, rather than a large number of repeated trials on the *same* system. This is an environment likely to be encountered in a project management situation.

The Poisson Theorem can be simplified to the following approximation for the probability of an event occurring k times in n trials,

$$\begin{aligned}\binom{n}{k} p^k q^{n-k} &= \frac{n!}{(n-k)!k!} \frac{(np)^k}{n^k} \left(1-\frac{np}{n}\right)^{n-k}, \\ &= \frac{\sqrt{(2\pi)e^{-n}n^{n+\frac{1}{2}}}}{\sqrt{(2\pi)(n-k)^{n-k+\frac{1}{2}}e^{-n+k}n^k}} \frac{(np)^k}{k!} e^{-np}, \\ &= \frac{1}{\left(1-\frac{k}{n!}\right)^n e^k} \frac{np^k}{k!} \\ &\approx \frac{np^k}{k!} e^{-np}.\end{aligned} \tag{1.6}$$

The exponential and Poisson expressions are directly related. A detailed understanding of this relationship will aid in the development of the analysis that follows.

Using the Poisson assumptions described in Table 1, the probability of n events prior to time t is,

$$P\{N = n | T \le t\} = P_t(n). \tag{1.7}$$

From of Eq. (1.7), the probability that no events occur $(n = 0)$ between time t and time $t + \Delta t$ is,

$$P_{t+\Delta t}(0) = P_t(0)[1 - \lambda \Delta t], \tag{1.8}$$

where the term $\lambda = np$ describing the total number of events is of moderate magnitude. The probability that n events occur between time t and time $t + \Delta t$ is then,

$$P_{t+\Delta t}(n) = P_t(n)[1 - \lambda \Delta t] + P_t(n-1)[\lambda \Delta t], \quad n > 0. \tag{1.9}$$

Using Eq. (1.9) and Eq. (1.8) and allowing $\Delta t \to 0$, a differential equation can be constructed describing the rate at which events occur between time t and time $t + \Delta t$,

$$\begin{aligned}\frac{d}{dt} P_t(0) &= -\lambda P_t(0), \\ \frac{d}{dt} P_t(n) &= \lambda \left[P_t(n-1) - P_t(n) \right], \text{ for } n > 0,\end{aligned} \tag{1.10}$$

with the initial conditions of,

$$P_t(n) = 0. \tag{1.11}$$

The unique solution to the differential equation in Eq. (1.10) is,

$$P_t(n) = \frac{(\lambda t)^n \, e^{-\lambda t}}{n!}, \quad n = 0, 1, 2, \ldots \tag{1.12}$$

which is the Poisson distribution defined in Eq. (1.6). Using Eq. (1.12) to define a function $F(t)$ representing the probability that no events have occurred as of time *t* gives,

$$F(t) = P_t\{n = 0\} = e^{-\lambda t}. \tag{1.13}$$

The expression in Eq. (1.13) is also the definition for the Cumulative Distribution Function, *CDF*, of the Poisson event process. By using Eq. (1.13), the probability distribution function, *pdf*, of the Poisson process can be given as,

$$f(t) = \lambda e^{-\lambda t}, \tag{1.14}$$

which is the exponential probability distribution. [2] The following statement describes the relationship between the Poisson and exponential expressions,

> *If the number of events occurring over an interval of time is Poisson distributed, then the time between events is exponentially distributed.*

An alternative method of relating the exponential and Poisson expressions is useful at this point. The functions defined in Eq. (1.1) and Eq. (1.2) are based on the interchangeability of the *pdf* and the *CDF* for any defined probability distribution. The Cumulative Distribution Function $F(x)$ of a random variable *X* is defined as a function obeying the following relationship,

$$F(x) = P\{X \le x\}, \quad -\infty < x < \infty. \tag{1.15}$$

2 This development of the *pdf* is very informal. Making use of the forward reference to construct an expression is circular logic and would not be permitted in more formal circumstances. For the purposes of this paper, this type of behavior can be tolerated, since the purpose of this development is to get to the results rather than dwell on the analysis process. This is a fundamental difference between mathematics and engineering.

The probability density function $f(x)$ of a random variable *X* can be derived from the *CDF* using the following,

$$f(x) = \frac{d}{dx} F(x). \tag{1.16}$$

The *CDF* can be obtained from the *pdf* by the following,

$$F(x) = P\{X \le x\} = \int_{-\infty}^{x} f(t)dt, \quad -\infty < x < \infty. \tag{1.17}$$

Using Eq. (1.16) and Eq. (1.17), the *CDF* and *pdf* expressions for an exponential distribution can be developed. If the mean time between events is an *Exponentially* distributed random variable, the *CDF* is,

$$F(t) = \begin{cases} 1 - e^{-\lambda t}, & 0 \le t \le \infty, \\ 0 \quad , & \text{otherwise}, \end{cases} \tag{1.18}$$

The number of events in the time interval $[0, t]$ is a *Poisson* distributed random variable with a probability density function of,

$$f(t) = \frac{d}{dt} F(t) = \begin{cases} \lambda e^{-\lambda}, & t > 0, \\ 0, & \text{otherwise}, \end{cases} \tag{1.19}$$

where *t* is a random variable denoting the time between events.

Conclusion

The original equation provided in the paper can be replaced with another that better describes the real world experiences of external events occurring that negatively impact the outcome of a project.

The probability of occurrence of such an event is not related to the simple linear passage of time, but rather to the parameters of the exponential distribution shown in Eq. (1.18). The simple phrase "the longer you wait the more likely something bad will happen," now has mathematical meaning.

Glen B. Alleman
Niwot Colorado
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